Back to the roots: rooting multivariate polynomials with numerical linear algebra and nD system theory

Prof. Dr. Bart De Moor

ESAT-STADIUS, KU Leuven, Kasteelpark Arenberg 10, 3001 Leuven, Belgium

T: +32(0)475287052 ; **E:** bart.demoor@esat.kuleuven.be ; **W:** www.bartdemoor.be

The achievements of *algebraic geometry* have been truly mind boggling, leading to Andrew Wiles' proof of Fermat's Last Theorem in number theory, or to Shing-Tung Yau's proof of the Calabi-Yau conjecture in string theory.

In this presentation, we return to the more prosaic 'original' roots of the field, which was born in the analysis of roots of univariate polynomials (Gauss' Fundamental Theorem of Linear Algebra, Galois' 'Impossibility' Theory). Later emphasis shifted to the analysis of systems of multivariate (i.e. in several unknowns) polynomial equations with integer, rational, real or complex coefficients. Many mathematicians (including e.g. Hilbert (the Hilbert Basis Theorem)) have contributed to the development of the field. In particular, ***finding the common roots of a system of multivariate polynomials with real coefficients is one of its key problems.*** The fact that the complexity of polynomial system solving is linked to the biggest challenges in mathematics (e.g. P=NP?, Hilbert's 10th Problem, etc.), attests to the depth of the problem. This key problem arises in an abundant number of applications in the sciences, mathematics and engineering (advanced geometry, chemical equilibrium systems, Nash equilibrium in game theory, cryptography, kinematics and robotics, Markov chain modeling, dynamical system modeling (structured total least squares), optimal control theory, etc.).

The last 20 years have witnessed the development of many new symbolic and numerical algorithms. The 'classical' approach to tackle the problem is based on the algebraic geometry framework of ideals, commutative rings and varieties, Euclidean division of multivariate polynomials and variations on the Buchberger algorithm to compute a Gröbner basis for the polynomial ideal. Other approaches can be labeled as 'hybrid', in which first a Gröbner basis is calculated using symbolic computations, followed by the numerical solution of one or more eigenvalue problems. Yet another school is using numerical homotopy methods. Recently, also relaxations of polynomial optimization problems have been considered, in which 19th century ideas (sums of squares and Hilbert's 17th problem, theory of moments) are blended with 20-21th century numerical semi-definite programming approaches.

However, the very nature of the problem is quite simple: Finding all the common roots of a system of multivariate polynomials is essentially equivalent to an eigenvalue problem in one variable (a fact known to Sylvester in the 19^{th} century). In this presentation, ***we will tackle the problem using multidimensional system theory and algorithms from numerical linear algebra***. We use the tools of numerical linear algebra with matrix decompositions (SVD and EVD), orthogonalization methods and multidimensional state space realization theory.